Communication Systems

Lecture 4

Signals and Systems

Correlation

- Because of the orthogonality, the two integrals of the cross product a zero
- This result can be extended to any number of mutually orthogonal signals

– Signal Comparison: Correlation

- Two vectors **g** and **x** are similar if **g** has a large component along **x**
  - In other words, if c is large, **g** and **x** are similar
  - A defective measure $c = \frac{g.x}{x.x} = \frac{1}{|x|^2} g.x$

– The amount of similarity between **g** and **x** should be independent of the lengths of **g** and **x**

- If we double the length of **g**, the amount of similarity between **g** and **x** should not change but here doubling **g** doubles the value of c, where as doubling **x** halves the value of c

Correlation

- Similarity between two vectors is indicated by the angle between the vectors
  - The smaller the angle, the larger is the similarity and vice versa

    The amount of similarity can therefore be conveniently measured by $\cos\theta$,

    A suitable measure would be $c_n = \cos\theta = \frac{\mathrm{g.x}}{|\mathrm{g}||\mathrm{x}|}$

    This measure is independent of the lengths of **g** and **x**

    This similarity measure is known as the correlation coefficient

    $$-1 \le c_n \le 1$$

    If the two vectors are aligned, the similarity is maximum $(c_n = 1)$

    Two vectors aligned in opposite direction have maximum dissimilarity $(c_n = -1)$

    If two vectors are orthogonal, the similarity is zero $(c_n = 0)$

Correlation

- Considering the signals over the entire time interval from $-\infty \text{ to } +\infty$ .To make c independent of the energies (sizes) of $g(t)$ and $x(t)$, we must normalize c by normalizing the two signals to have unit energies

  Thus the appropriate similarity index c_n is given by

$$c_n = \frac{1}{\sqrt{E_x E_g}} \int_{-\infty}^{\infty} g(t)x(t)dt$$

  Multiplying $g(t)$ or $x(t)$ by any constant has no effect on this index thus it is independent of the sizes of $g(t)$ and $x(t)$

  Schwarz inequality states that for two real signal $g(t)$ and $x(t)$

$$\left[\int_{-\infty}^{\infty} g(t)x(t)dt\right]^2 \leq E_g E_x$$

  thus

$$-1 \leq c_n \leq 1$$

Correlation

- If $g(t) = Kx(t)$, then $c_n = 1$ when K is a positive constant and $c_n = -1$ when K is any negative constant and $c_n = 0$ if $g(t)$ and $x(t)$ are orthogonal

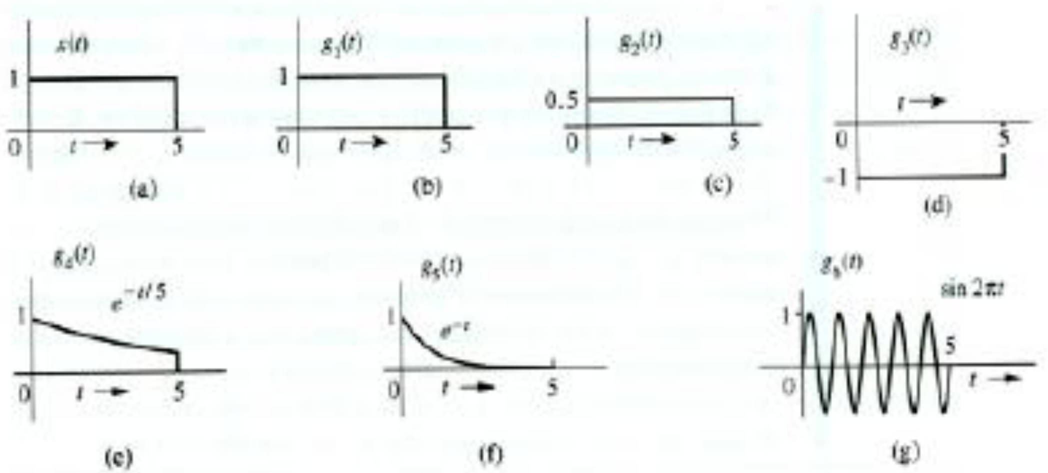
  Thus the maximum similarity $[g(t) = Kx(t)]$ is indicated by $c_n = 1$, the maximum dissimilarity $[g(t) = -Kx(t)]$ is indicated by $c_n = -1$

  and when two signals are orthogonal, the similarity is zero
- We may view orthogonal signals as unrelated signals
  - Maximum dissimilarity is different from unrelatedness
  - We have the best friends $c_n = 1$, worst enemies $c_n = -1$ and complete strangers $c_n = 0$
- For complex signals

$$c_n = \frac{1}{\sqrt{E_x E_g}} \int_{-\infty}^{\infty} g(t) x^*(t) dt$$

Correlation

- Example
  - Find the correlation between $x(t)$ and $g_i(t), i = 1,2,3,4,5,6$



Application to signal detection

- Correlation measures the degree of similarity between two signals
  - Consider the case of binary communication, where two known waveforms are received in a random sequence
  - Each time we receive a pulse, our task is to determine which of the two (known) waveforms is received
  - To make the detection easier, we must make the two pulses as dissimilar as possible, which means that we should select one pulse as the negative of the other pulse. This gives the highest dissimilarity $c_n = -1$
    - This scheme is known as antipodal scheme
  - We can use orthogonal pulses which result in $c_n = 0$
  - In practice both these options are used, although antipodal one is the best in terms of distinguishability between the two pulses

Application to signal detection

- Let us consider the antipodal scheme in which the two pulses are $p(t)$ and $-p(t)$

  The correlation coefficient c_n of these pulses is -1

  The receiver consists of a correlator that computes the correlation between $p(t)$ and the received pulse

  If the correlation is 1, we decide that $p(t)$ is received

  If the correlation is -1, we decide that $-p(t)$ is received

  - Due to channel noise and distortion, the correlation coefficient has a smaller magnitude thus reducing the distinguishability
  - We use a threshold detector, which decides that if the correlation is positive, the received pulse is $p(t)$ and if the correlation is negative, the received pulse is $-p(t)$

Application to signal detection

- Application of correlation to signal detection in a radar
  - A signal pulse is transmitted in order to detect a suspected target and if a target is present the pulse will be reflected by it
  - By measuring the time delay between the transmitted and received pulses (reflected) pulses we determine the distance of the target
  - Let the transmitted and reflected pulses be $g(t)$ and $z(t)$ respectively

$$c_n = \frac{1}{\sqrt{E_x E_g}} \int_{-\infty}^{\infty} g(t)x(t)dt = 0$$

    - Because the two pulses are disjoint (nonoverlapping in time)
  - To avoid this difficulty, we compare the transmitted pulse with the received pulse, shifted by τ

Correlation Functions

- If for some value of τ , there is a strong correlation, we not only detect the presence of the pulse but we also detect the relative time shift of $z(t)$ with respect to $g(t)$
- For this reason we use the modified integral $\psi_{gz}(\tau)$, the cross-correlation function of two real signals $g(t)$ and $z(t)$ defined by

$$\psi_{gz}(\tau)=\int_{-\infty}^{\infty} g(t)z(t+\tau)dt$$

- Autocorrelation function
  - The correlation of a function with itself is called autocorrelation
  - The autocorrelation function $\psi_g(\tau)$ of a real signal $g(t)$ is defined as:

$$\psi_g(\tau)=\int_{-\infty}^{\infty} g(t)z(t+\tau)dt$$

  - Autocorrelation function provides a valuable spectral informatio

Autocorrelation Example

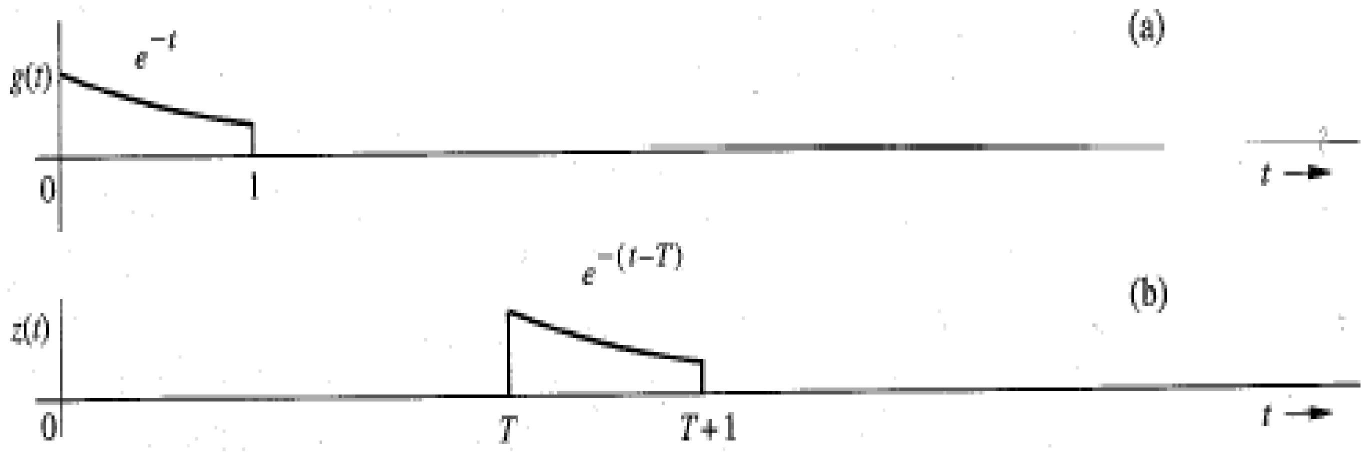


Figure 2.19 Physical explanation of the autocorrelation function.

Signal representation by orthogonal set

- Representing a signal as sum of orthogonal signals
  - Insight can be gained from a similar problem with vectors
  - A vector can be represented as sum of orthogonal vectors, which form the coordinate system of a vector space
- Orthogonal vector space
  - Consider a three dimensional Cartesian vector space, described by three mutually orthogonal vectors x_1, x_2 and x_3
  - Approximate a three dimensional vector g in terms of two mutually orthogonal vectors x_1 and x_2

$$g \cong c_1 x_1 + c_2 x_2$$

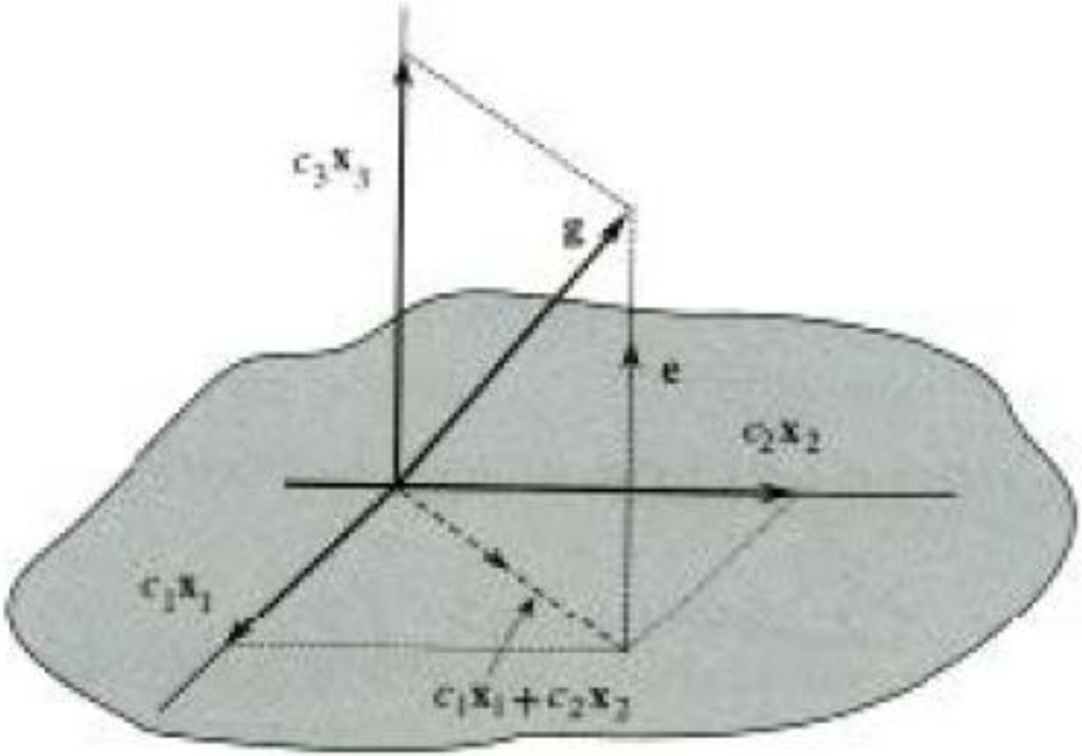
The error in this approximation is

$$e = g - (c_1 x_1 + c_2 x_2)$$

$$g = c_1 x_1 + c_2 x_2 + e$$

Signal representation by orthogonal set

– Representation of a vector in three dimensional space



Signal representation by orthogonal set

- The length of **e** is minimum when **e** is perpendicular to the $x_1 - x_2$ and $c_1 x_1$ and $c_2 x_2$ are the projections (components) of g on x_1 and x_2 respectively
  Where the constants c_1 and c_2 are given by $c = \frac{\mathrm{g.x}}{\mathrm{x.x}} = \frac{1}{|\mathrm{x}|^2}\mathrm{g.x}$

- Now the best approximation to g in terms of all the three mutually orthogonal vectors x_1 , x_2 and x_3 :

$$\mathrm{g} \cong c_1 x_1 + c_2 x_2 + c_3 x_3$$

A unique choice of c_1, c_2 and c_3 exist for which the above approximation becomes an equality

$$\mathrm{g} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

- In this case $c_1\mathbf{x}_1$, $c_2\mathbf{x}_2$ and $c_3\mathbf{x}_3$ are projections of **g** on $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, respectively

Signal representation by orthogonal set

- The error in the approximation is zero when **g** is approximated in terms of three mutually orthogonal vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$
  - This is because **g** is a three dimensional vector and the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ represent a complete set of orthogonal vectors in three dimensional space
  - **Completeness** here means that it is impossible to find in this space another vector $\mathbf{x}_4$, that is orthogonal to all three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$
  - Any vector in this space can then be represented (with zero error) in terms of these three vectors. Such vectors are known as basis vectors
- If a set of vectors $\{x_i\}$ is not complete, the error in approximation will generally not be zero
  - Thus in a three dimensional case, it is generally not possible to represent a vector **g** in terms of only two basis vectors, without an error

– The choice of basis vectors is not unique and infact a set of basis vectors corresponds to a particular coordinate system

Signal representation by orthogonal set

- A three dimensional vector **g** may be represented in many different ways, depending on the coordinate system used
  - If a set of vectors $\{x_i\}$ is mutually orthogonal, that is, if

$$x_m . x_n = \begin{cases} 0 & m \neq n \\ |x_m|^2 & m = n \end{cases}$$

  - If this basis set is complete, a vector **g** in this space can be expressed as

$$\mathrm{g} = c_1 \, \mathrm{x}_1 + c_1 \, \mathrm{x}_2 + c_3 \, \mathrm{x}_3$$

  - Where the constants c_i are given by

$$c = \frac{\mathrm{g} \, . \, \mathrm{x}_i}{\mathrm{x}_i \, . \, \mathrm{x}_i} = \frac{1}{|\mathrm{x}_i|^2} \mathrm{g} \, . \, \mathrm{x}_i, \qquad i = 1,2,3$$

Orthogonal Signal Space

- Orthogonal Signal Space
  - The orthogonality of a signal set $x_1(t), x_2(t), \ldots, x_N(t)$ over the interval $[t_1, t_2]$

$$\int_{t_1}^{t_2} x_m(t)x_n(t)dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

  - If the energies $E_n = 1$ for all n, then the set is normalized and is called an orthonormal set
  - An orthogonal set can always be normalized by dividing $x_n(t)$ by $\sqrt{E_n}$ for all n
- Consider approximating a signal $g(t)$ over the interval $[t_1, t_2]$ by a set of N mutually orthogonal signals $x_1(t), x_2(t), \ldots, x_N(t)$

$$g(t) \cong c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t)$$

Orthogonal signal space

$$g(t) \cong c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t)$$

$$= \sum_{n=1}^{N} c_n x_n(t) \qquad t_1 \le t \le t_2$$

– The energy of error signal is minimized if we choose

$$c_n = \frac{\int_{t_1}^{t_2} g(t) x_n(t) dt}{\int_{t_1}^{t_2} x^2{}_n(t) dt}$$

$$= \frac{1}{E_n} \int_{t_1}^{t_2} g(t) x_n(t) dt \qquad n = 1,2,3 \cdots N$$

- Moreover if the orthogonal set is **Complete**, the error energy approaches zero $\rightarrow 0$

Orthogonal signal space

- And the approximation is now an equality

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$

$$= \sum_{n=1}^{\infty} c_n x_n(t) \qquad t_1 \le t \le t_2$$

- Because the error energy approaches zero, it follows that the energy of $g(t)$ is now equal to the sum of the energies of it's orthogonal components $c_1 x_1(t), c_2 x_2(t), c_3 x_3(t), \cdots$
- The series on the right hand side is called the generalized Fourier series of $g(t)$ with respect to the set $\{x_n(t)\}$
- When the set $\{x_n(t)\}$ is such that the error energy $E_e \to 0$ as $N \to \infty$ for every member of some particular class, we say that the set $\{x_n(t)\}$ is complete on $[t_1, t_2]$ for that class of $g(t)$, and the set $\{x_n(t)\}$ is called the set of **basis functions** or **basis signals**

Example

- Approximation of rectangular function by orthogonal function
- In example:2.5, Page 33, a rectangular function g(t) is approximated by sin(t) by $g(t) \cong \frac{4}{\pi} \sin t$
- Functions of type sin(nwt) and sin(mwt) are mutually orthogonal over one period

$$g(t) \cong C_1 \sin t + C_2 \sin 2t + \dots\dots + C_n \sin nt$$

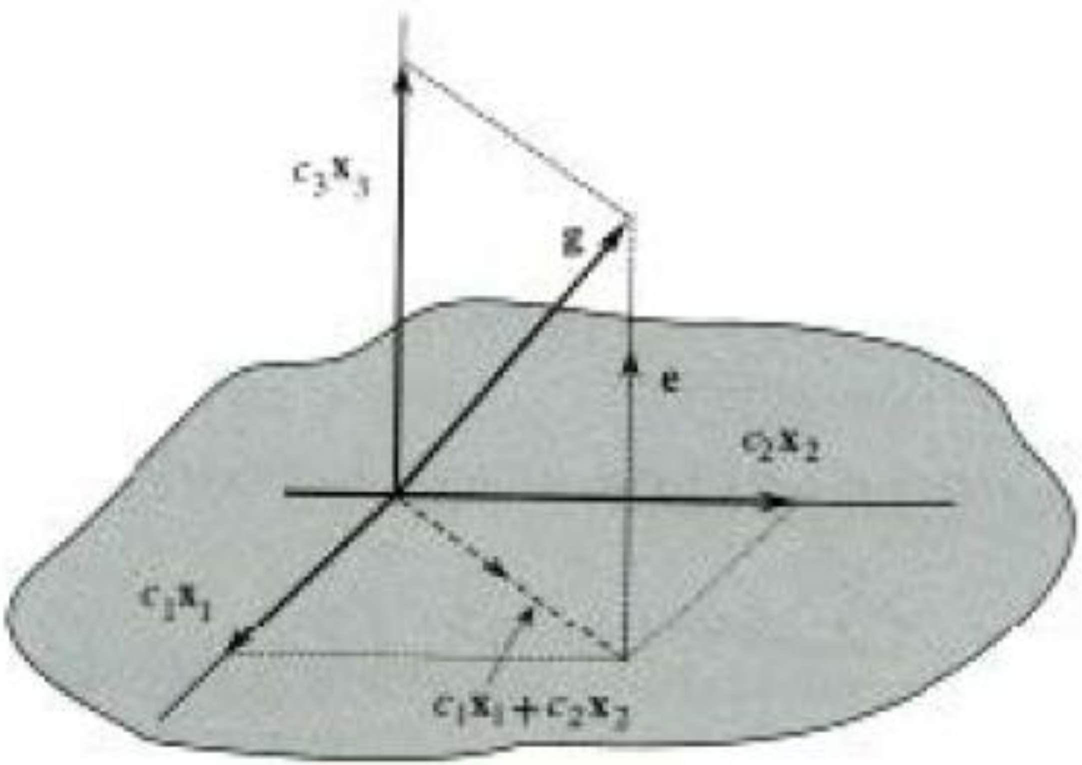
$$C_n = \frac{4}{\pi n}, n = odd$$

$$C_n = 0, n = even$$

$$g(t) \cong \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \dots\dots\right)$$

Signal representation by orthogonal set

- Representation of a vector in three dimensional space



Signal representation by orthogonal set

- Now the best approximation to $\mathbf{g}$ in terms of all the three mutually orthogonal vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$\mathrm{g} \cong c_1 x_1 + c_2 x_2 + c_3 x_3$$

  A unique choice of c_1, c_2 and c_3 exist for which the above approximation becomes an equality

$$\mathrm{g} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

- In this case $c_1\mathbf{x}_1$, $c_2\mathbf{x}_2$ and $c_3\mathbf{x}_3$ are projections of $\mathbf{g}$ on $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, respectively

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$$\mathrm{g} = c_1 \mathrm{x}_1 + c_1 \mathrm{x}_2 + c_3 \mathrm{x}_3$$

  - Where the constants c_i are given by

$$c = \frac{\mathrm{g} . \mathrm{x}_i}{\mathrm{x}_i . \mathrm{x}_i} = \frac{1}{|\mathrm{x}_i|^2} \mathrm{g} . \mathrm{x}_i, \qquad i = 1,2,3$$

Orthogonal signal space

$$g(t) \cong c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t)$$

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Orthogonal signal space

- And the approximation is now an equality

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$

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Trigonometric Fourier Series

Consider a signal set:

$$\{1, \cos \omega_0 t, \cos 2\omega_0 t, \ldots, \cos n\omega_0 t, \ldots, \sin \omega_0 t, \sin 2\omega_0 t, \ldots, \sin n\omega_0 t, \ldots\} \tag{2.60}$$

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & m = n \neq 0 \end{cases} \tag{2.61a}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases} \tag{2.61b}$$

and

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0 \qquad \text{for all } n \text{ and } m \tag{2.61c}$$

$$g(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots \qquad t_1 \le t \le t_1 + T_0 \tag{2.62a}$$

or

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad t_1 \le t \le t_1 + T_0 \tag{2.62b}$$

where

$$\omega_0 = \frac{2\pi}{T_0} \tag{2.63}$$

Using Eq. (2.57), we can determine the Fourier coefficients a_0, a_n, and b_n. Thus,

$$a_n = \frac{\int_{t_1}^{t_1+T_0} g(t) \cos n\omega_0 t \, dt}{\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t \, dt} \tag{2.64}$$

The integral in the denominator of Eq. (2.64) as seen from Eq. (2.61a) (with $m = n$) is $T_0/2$ when $n \neq 0$. Moreover, for $n = 0$, the denominator is T_0. Hence,

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} g(t) \, dt \tag{2.65a}$$

and

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cos n\omega_0 t \, dt \qquad n = 1, 2, 3, \ldots \tag{2.65b}$$

Using a similar argument, we obtain

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \sin n\omega_0 t \, dt \qquad n = 1, 2, 3, \ldots \tag{2.65c}$$

Compact Trigonometric Fourier Series

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n) \tag{2.66}$$

where

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{2.67a}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \tag{2.67b}$$

For consistency we denote the dc term a_0 by C_0, that is,

$$C_0 = a_0 \tag{2.67c}$$

Using the identity (2.66), the trigonometric Fourier series in Eq. (2.62) can be expressed in the **compact form** of the trigonometric Fourier series as

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad t_1 \le t \le t_1 + T_0 \tag{2.68}$$

Example 2.7 **Find the compact trigonometric Fourier series for the exponential $e^{-t/2}$ shown in Fig 2.21a over the interval $0 \le t \le \pi$**

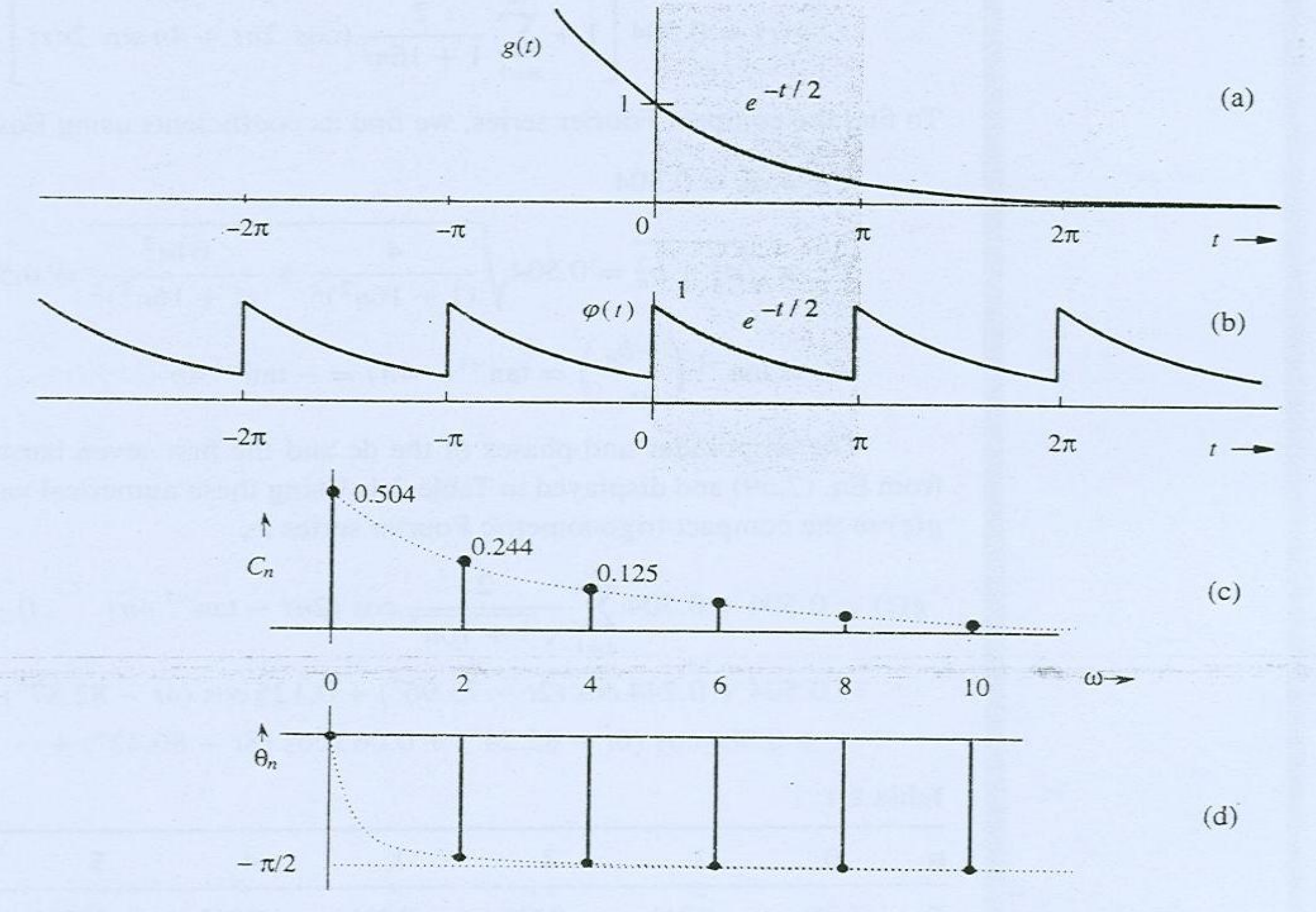

Figure 2.21 Periodic signal and its Fourier spectra.

Because we are required to represent $g(t)$ by the trigonometric Fourier series over the interval $0 \le t \le \pi$, $T_0 = \pi$, and the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T_0} = 2 \text{ rad/s}$$

Therefore,

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + b_n \sin 2nt \qquad 0 \le t \le \pi$$

where [from Eq. (2.65a)]

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\, dt = 0.50$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2} \cos 2nt\, dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2} \sin 2nt\, dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

Therefore,

$$g(t) = 0.504\left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}(\cos 2nt + 4n \sin 2nt)\right] \qquad 0 \le t \le \pi$$

$$C_0 = a_0 = 0.504$$

$$C_n = \sqrt{a_n^2 + b_n^2} = 0.504\sqrt{\frac{4}{(1+16n^2)^2} + \frac{64n^2}{(1+16n^2)^2}} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}(-4n) = -\tan^{-1} 4n \qquad (2.69)$$

The amplitudes and phases of the dc and the first seven harmonics are computed from Eq. (2.69) and displayed in Table 2.1. Using these numerical values, we can express $g(t)$ in the compact trigonometric Fourier series as

$$g(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos\left(2nt - \tan^{-1} 4n\right) \qquad 0 \le t \le \pi \qquad (2.70a)$$

$$= 0.504 + 0.244\cos(2t - 75.96^\circ) + 0.125\cos(4t - 82.87^\circ)$$
$$+ 0.084\cos(6t - 85.24^\circ) + 0.063\cos(8t - 86.42^\circ) + \cdots \quad 0 \le t \le \pi \quad (2.70b)$$

Table 2.1

n	0	1	2	3	4	5	6	7
C_n	0.504	0.244	0.125	0.084	0.063	0.0504	0.042	0.036
θ_n	0	−75.96	−82.87	−85.24	−86.42	−87.14	−87.61	−87.95

Fourier Spectrum

- Amplitude Spectrum
- Phase Spectrum
- Frequency Spectra

There are two basic conditions for the existence of the Fourier series.

1. For the series to exist, the coefficients a_0, a_n, and b_n in Eqs. (2.65) must be finite. From Eqs. (2.65) it follows that the existence of these coefficients is guaranteed if $g(t)$ is absolutely integrable over one period; that is,

$$\int_{T_0} |g(t)|\, dt < \infty \tag{2.73}$$

$$\int_{T_0} |g(t)|\, dt < \infty \tag{2.73}$$

2. The function $g(t)$ have only a finite number of maxima and minima in one period, and it may have only a finite number of finite discontinuities in one period.

These two conditions are known as the **strong Dirichlet conditions**. We note here that any periodic waveform that can be generated in a laboratory satisfies strong Dirichlet **conditions.**

$$a_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \cos\, n\omega_0 t\, dt = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n = 1, 5, 9, 13, \cdots \\ -\dfrac{2}{\pi n} & n = 3, 7, 11, 15, \cdots \end{cases} \qquad (2.74b)$$

$$b_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \sin\, nt\, dt = 0 \qquad (2.74c)$$

In these derivations we used the fact that $\omega_0 T_0 = 2\pi$. Therefore,

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\, \omega_0 t - \frac{1}{3}\cos\, 3\omega_0 t + \frac{1}{5}\cos\, 5\omega_0 t - \frac{1}{7}\cos\, 7\omega_0 t + \cdots\right) \qquad (2.75)$$

$$-\cos\, x = \cos\,(x - \pi)$$

Using this fact, we can express the series in Eq. (2.75) as

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\, \omega_0 t + \frac{1}{3}\cos\,(3\omega_0 t - \pi) + \frac{1}{5}\cos\, 5\omega_0 t + \frac{1}{7}\cos\,(7\omega_0 t - \pi) + \frac{1}{9}\cos\, 9\omega_0 t + \cdots\right]$$

This is the desired form of the compact trigonometric Fourier series. The amplitudes are

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3, 7, 11, 15, \cdots \\ -\pi & n = 3, 7, 11, 15, \cdots \end{cases}$$

Corollary

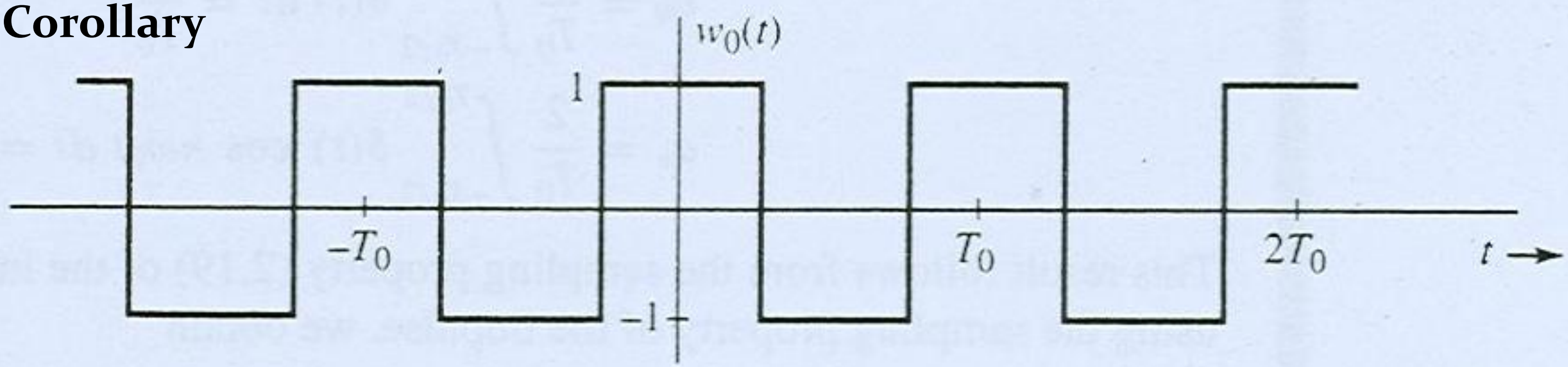

Figure 2.23 Bipolar square pulse periodic signal.

Another useful function that is related to the periodic square wave is the bipolar square wave $w_0(t)$ shown in Fig. 2.23a. We encounter this signal in switching applications. Note that $w_0(t)$ is basically $w(t)$ minus its dc component. It is easy to see that

$$w_0(t) = 2[w(t) - 0.5]$$

Hence, from Eq. (2.75) it follows that

$$w_0(t) = \frac{4}{\pi}\left(\cos\ \omega_0 t - \frac{1}{3}\cos\ 3\omega_0 t + \frac{1}{5}\cos\ 5\omega_0 t - \frac{1}{7}\cos\ 7\omega_0 t + \cdots\right) \quad (2.76)$$

EXAMPLE 2.9 Find the trigonometric Fourier series and sketch the corresponding spectra for the periodic impulse train $\delta_{T_0}(t)$ shown in Fig. 2.24a.

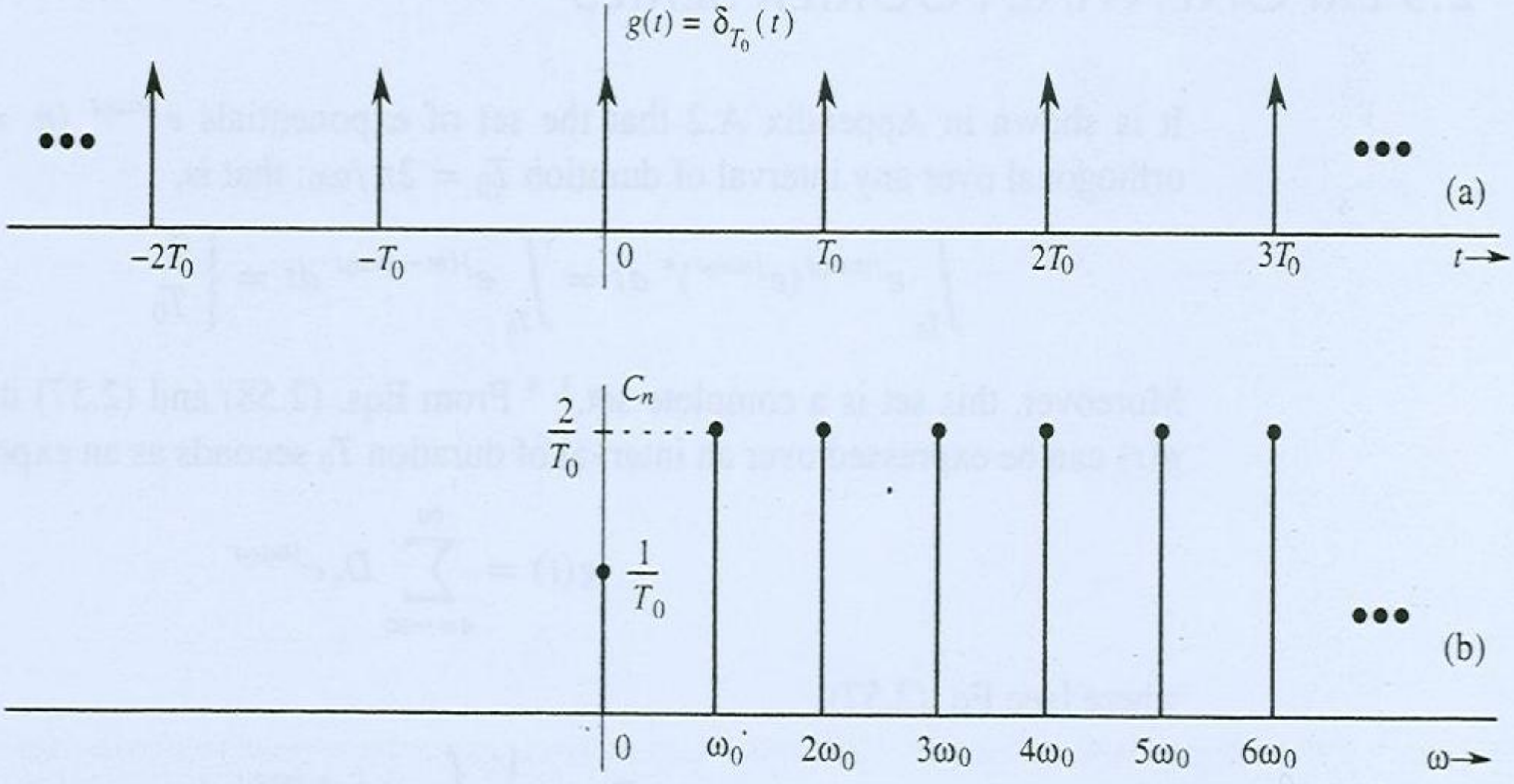

Figure 2.24 Impulse train and its Fourier spectrum.

The trigonometric Fourier series for $\delta_{T_0}(t)$ is given by

$$\delta_{T_0}(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n) \qquad \omega_0 = \frac{2\pi}{T_0}$$

We first compute a_0, a_n, and b_n:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, dt = \frac{1}{T_0}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \cos n\omega_0 t\, dt = \frac{2}{T_0}$$

This result follows from the sampling property (2.19) of the impulse function. Similarly, using the sampling property of the impulse, we obtain

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \sin n\omega_0 t\, dt = 0$$

Therefore, $C_0 = 1/T_0$, $C_n = 2/T_0$, and $\theta_n = 0$. Thus,

$$\delta_{T_0}(t) = \frac{1}{T_0}\left(1 + 2\sum_{n=1}^{\infty} \cos n\omega_0 t\right) \tag{2.77}$$

Figure 2.24b shows the amplitude spectrum. The phase spectrum is zero.

Exponential Fourier Series

It is shown in Appendix A.2 that the set of exponentials $e^{jn\omega_0 t}$ $(n = 0, \pm 1, \pm 2, \ldots)$ is orthogonal over any interval of duration $T_0 = 2\pi/\omega_0$; that is,

$$\int_{T_0} e^{jm\omega_0 t}(e^{jn\omega_0 t})^* \, dt = \int_{T_0} e^{j(m-n)\omega_0 t} \, dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases} \tag{2.78}$$

Moreover, this set is a complete set.[3, 4] From Eqs. (2.58) and (2.57) it follows that a signal $g(t)$ can be expressed over an interval of duration T_0 seconds as an exponential Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{2.79}$$

where [see Eq. (2.57)]

$$D_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_0 t} \, dt \tag{2.80}$$

$$\begin{aligned} C_n \cos(n\omega_0 t + \theta_n) &= \frac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right] \\ &= \left(\frac{C_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \left(\frac{C_n}{2} e^{-j\theta_n}\right) e^{-jn\omega_0 t} \\ &= D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \end{aligned} \tag{2.81}$$

where

$$\begin{aligned} D_n &= \frac{1}{2} C_n e^{j\theta_n} \\ D_{-n} &= \frac{1}{2} C_n e^{-j\theta_n} \end{aligned} \tag{2.82}$$

The compact trigonometric Fourier series of a periodic signal $g(t)$ is given by

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

The use of Eq. (2.81) in the preceding equation (and letting $C_0 = D_0$) yields

$$g(t) = D_0 + \sum_{n=1}^{\infty} D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}$$

$$= D_0 + \sum_{n=-\infty\ (n\neq 0)}^{\infty} D_n e^{jn\omega_0 t}$$

EXAMPLE 2.10 Find the exponential Fourier series for the signal in Fig. 2.21b (Example 2.7).

In this case, $T_0 = \pi$, $\omega_0 = 2\pi/T_0 = 2$, and

$$\varphi(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

where

$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{T_0} \varphi(t)e^{-j2nt}\,dt \\ &= \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,e^{-j2nt}\,dt \\ &= \frac{1}{\pi}\int_0^{\pi} e^{-(\frac{1}{2}+j2n)t}\,dt \\ &= \frac{-1}{\pi\left(\frac{1}{2}+j2n\right)} e^{-(\frac{1}{2}+j2n)t}\Big|_0^{\pi} \\ &= \frac{0.504}{1+j4n} \end{aligned} \tag{2.83}$$

and

$$\varphi(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1+j4n} e^{j2nt} \tag{2.84a}$$

$$\begin{aligned} &= 0.504\left[1 + \frac{1}{1+j4}e^{j2t} + \frac{1}{1+j8}e^{j4t} + \frac{1}{1+j12}e^{j6t} + \cdots \right. \\ &\quad \left. + \frac{1}{1-j4}e^{-j2t} + \frac{1}{1-j8}e^{-j4t} + \frac{1}{1-j12}e^{-j6t} + \cdots\right] \end{aligned} \tag{2.84b}$$

Observe that the coefficients D_n are complex. Moreover, D_n and D_{-n} are conjugates, as expected [see Eqs. (2.82)].

Exponential Fourier Spectra

$$|D_n| = |D_{-n}| = \frac{1}{2}C_n \quad (2.85a)$$

$$\angle D_n = \theta_n \quad \text{and} \quad \angle D_{-n} = -\theta_n \quad (2.85b)$$

Thus,

$$D_n = |D_n|e^{j\theta_n} \quad \text{and} \quad D_{-n} = |D_n|e^{-j\theta_n} \quad (2.86)$$

Note that $|D_n|$ are the amplitudes (magnitudes) and $\angle D_n$ are the angles of various exponential components. From Eqs. (2.85) it follows that the amplitude spectrum ($|D_n|$ vs. ω) is an even function of ω and the angle spectrum ($\angle D_n$ vs. ω) is an odd function of ω when $g(t)$ is a real signal.

For the series in Example 2.10 [Eq. (2.84b)], for instance,

$$D_0 = 0.504$$

$$D_1 = \frac{0.504}{1+j4} = 0.122e^{-j75.96^\circ} \Longrightarrow |D_1| = 0.122 \qquad \angle D_1 = -75.96^\circ$$

$$D_{-1} = \frac{0.504}{1-j4} = 0.122e^{j75.96^\circ} \Longrightarrow |D_{-1}| = 0.122 \qquad \angle D_{-1} = 75.96^\circ$$

and

$$D_2 = \frac{0.504}{1+j8} = 0.0625e^{-j82.87^\circ} \Longrightarrow |D_2| = 0.0625 \qquad \angle D_2 = -82.87^\circ$$

$$D_{-2} = \frac{0.504}{1-j8} = 0.0625e^{j82.87^\circ} \Longrightarrow |D_{-2}| = 0.0625 \qquad \angle D_{-2} = 82.87^\circ$$

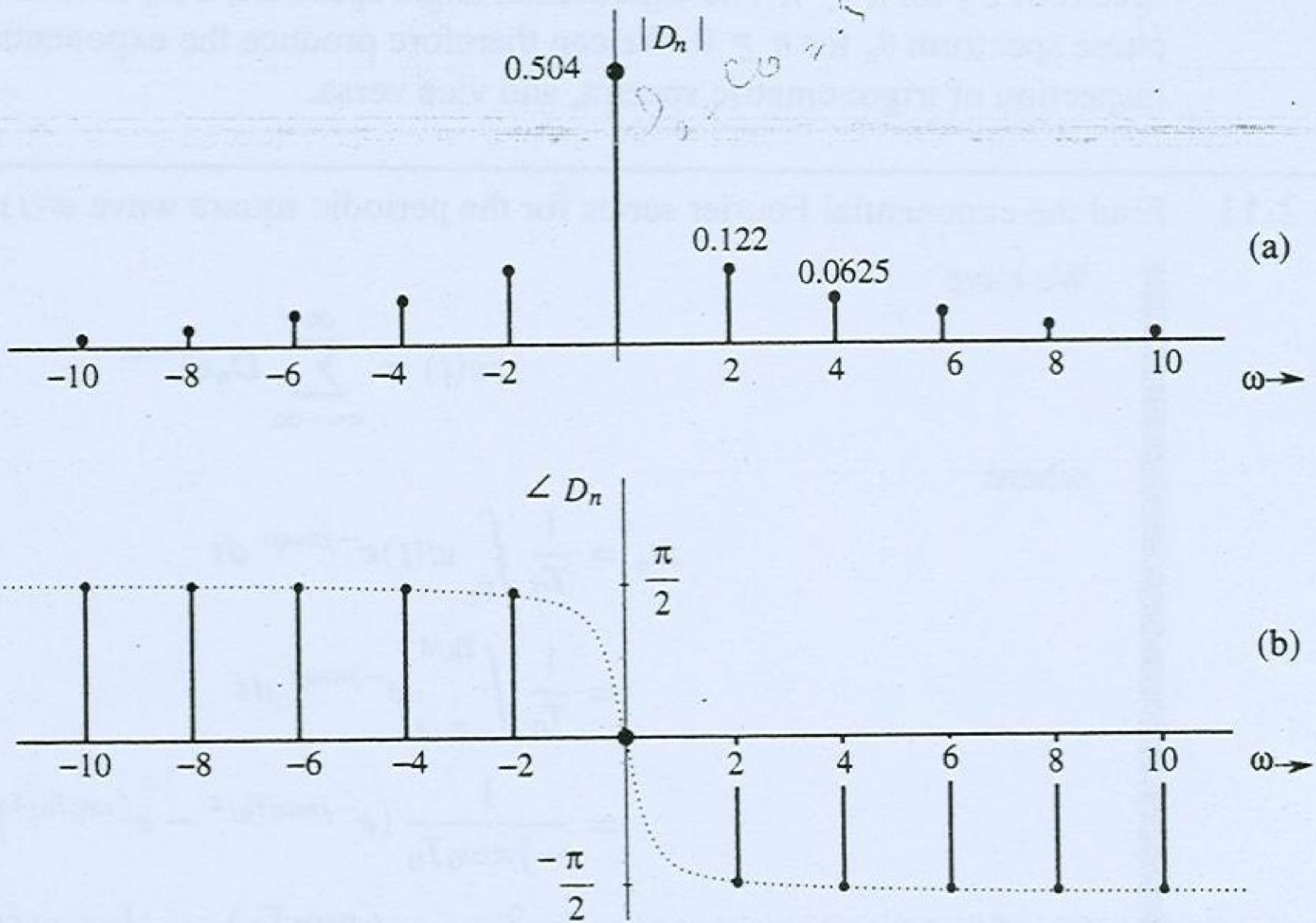

Figure 2.25 Exponential Fourier spectra for the signal in Fig. 2.21a.

Example 2.11

Find the exponential Fourier series for the periodic square wave $w(t)$ shown in Fig. 2.22a.

We have

$$w(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

$$D_n = \frac{1}{T_0} \int_{T_0} w(t) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{-jn\omega_0 T_0} \left(e^{-jn\omega_0 T_0/4} - e^{jn\omega_0 T_0/4} \right)$$

$$= \frac{2}{n\omega_0 T_0} \sin\left(\frac{n\omega_0 T_0}{4}\right) = \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

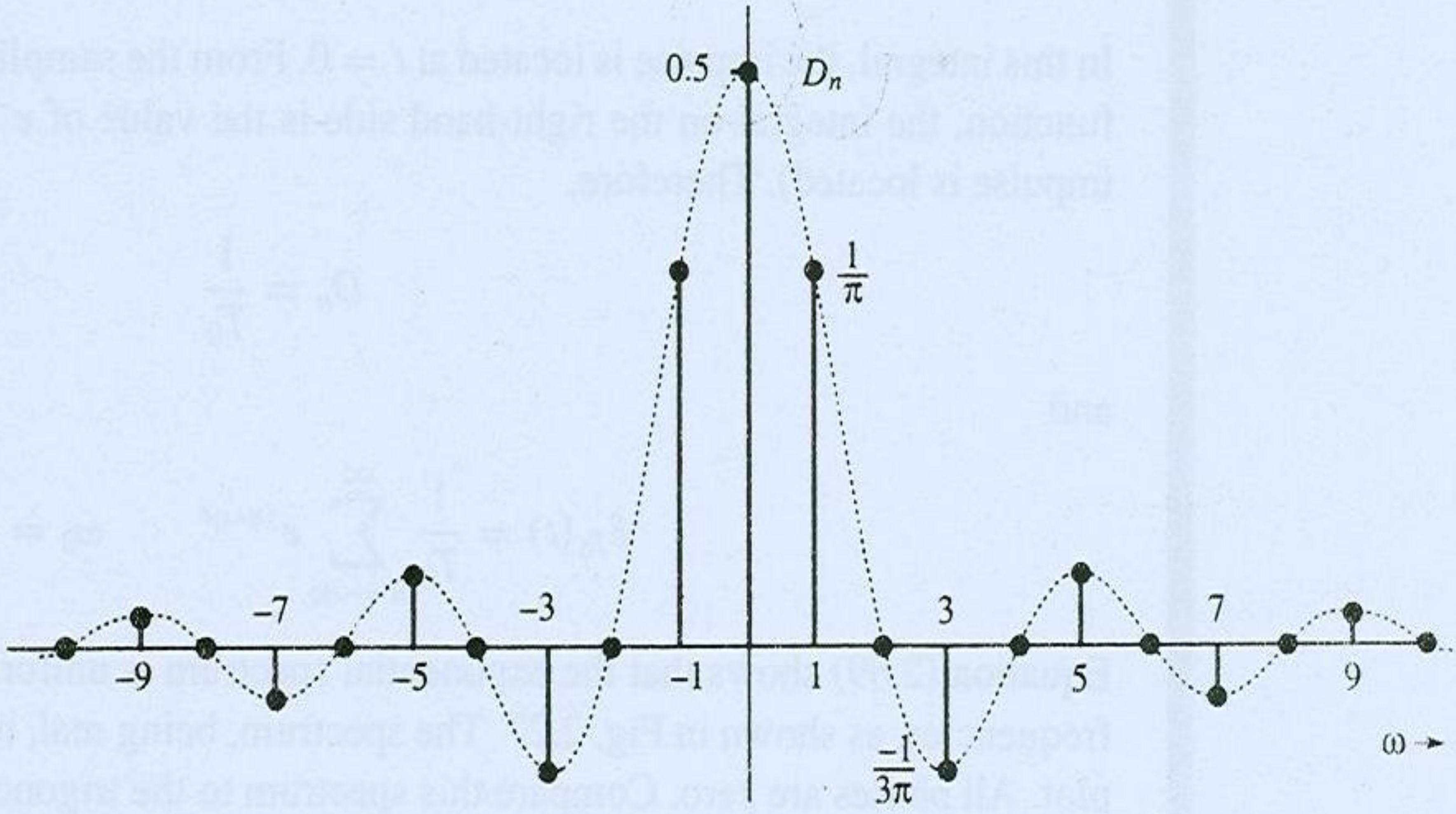

Figure 2.26 Exponential Fourier spectrum of the square pulse periodic signal.

EXAMPLE 2.12 Find the exponential Fourier series and sketch the corresponding spectra for the impulse train $\delta_{T_0}(t)$ shown in Fig. 2.27.

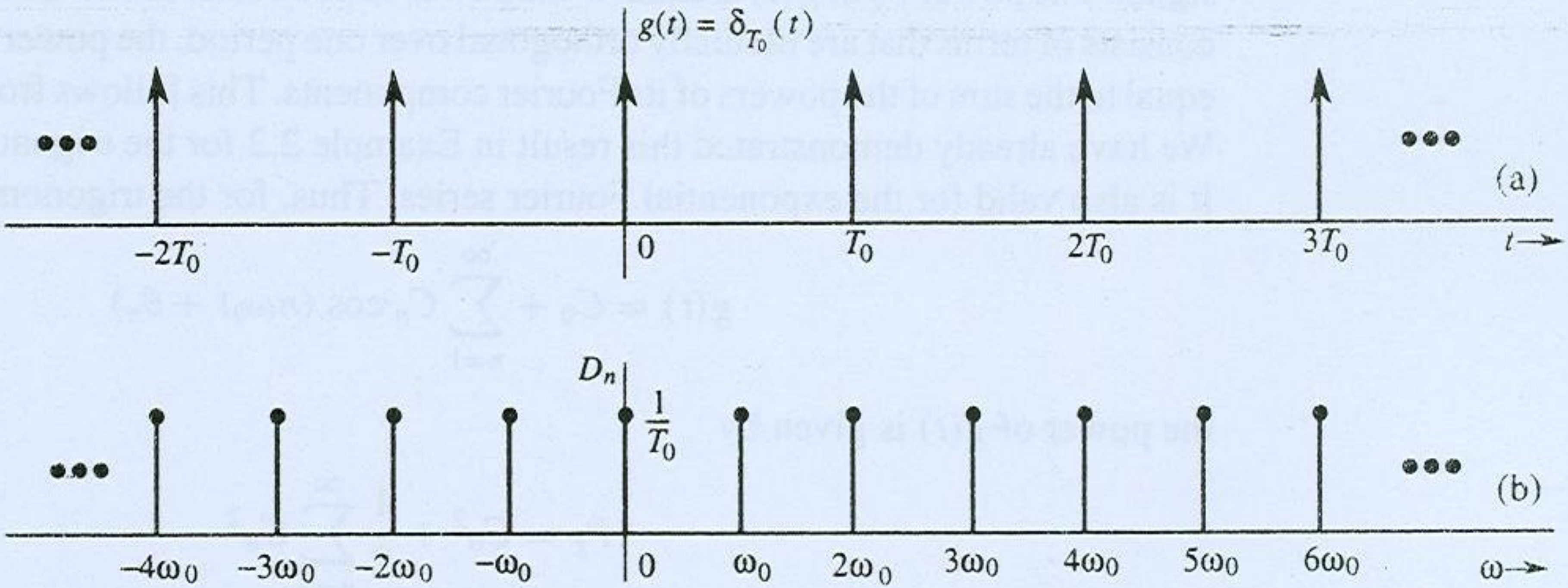

Figure 2.27 Impulse train and its exponential Fourier spectra.

The exponential Fourier series is given by

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{2.87}$$

where

$$D_n = \frac{1}{T_0} \int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t}\, dt$$

Choosing the interval of integration $(-T_0/2,\ T_0/2)$ and recognizing that over this interval $\delta_{T_0}(t) = \delta(t)$,

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t}\, dt$$

In this integral, the impulse is located at $t = 0$. From the sampling property of the impulse function, the integral on the right-hand side is the value of $e^{-jn\omega_0 t}$ at $t = 0$ (where the impulse is located). Therefore,

$$D_n = \frac{1}{T_0} \tag{2.88}$$

and

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{2.89}$$

Parseval's theorem

- Parseval's theorem
  - Energy of a signal is equal to sum of energies of its orthogonal components

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$

$$E_g = c_1^{\ 2} E_1 + c_2^{\ 2} E_2 + c_3^{\ 2} E_3 + \cdots$$

$$= \sum_n c_n^{\ 2} E_n$$

Parseval's Theorem

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

the power of $g(t)$ is given by

$$P_g = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \qquad (2.90)$$

For the exponential Fourier series

$$g(t) = D_0 + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} D_n e^{jn\omega_0 t}$$

the power is given by (see Prob. 2.1-7)

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2 \qquad (2.91a)$$

For a real $g(t)$, $|D_{-n}| = |D_n|$. Therefore,

$$P_g = D_0^2 + 2\sum_{n=1}^{\infty} |D_n|^2 \qquad (2.91b)$$